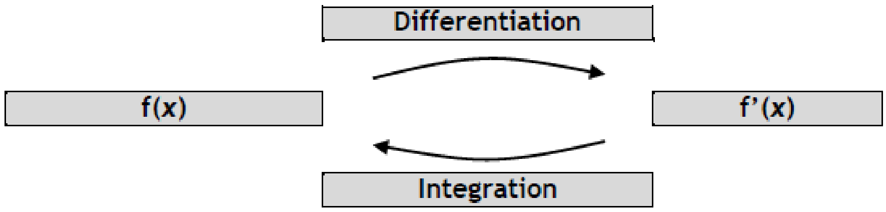
Calculus 2: Integration

The reverse process to differentiation is known as **integration**.



As it is the opposite of finding the derivative, the function obtained by integration is sometimes called the **anti-derivative**, but is more commonly known as the **integral**, and is given the sign $\int$.

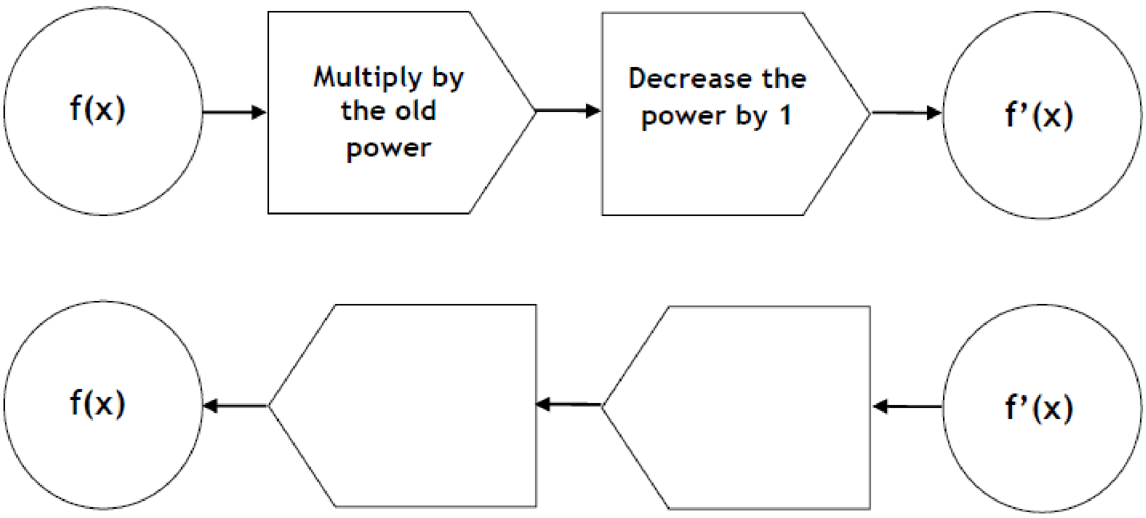
If f(x) = x^n, then $\int x^n dx$ is "the integral of x^n with respect to x "

Indefinite Integrals and the Constant of Integration

Consider the three functions $a(x) = 3x^2 + 2x + 5$, $b(x) = 3x^2 + 2x - 8$ and $c(x) = 3x^2 + 2x - \frac{13}{4}$.

In each case, the derivative of the function is the same, i.e. $6x + 2$. This means that $\int (6x+2)dx$ has more than one answer. Because there is more than one answer, we say that this is an **indefinite integral**, and we must include in the answer a constant value C, to represent the 5, -8, $-\frac{13}{4}$ etc which we would need to distinguish a(x) from b(x) from c (x) etc.

To find the integral of a function, we do the opposite of what we would do to find the derivative:



In general:

$$\int ax^n dx = \frac{ax^{n+1}}{n+1} + C \quad (n \neq -1)$$

INtegration ***IN***creases the power!

1. Write as ax^n
2. Increase the power by 1
3. Divide by the new power

Example 1: Find (remember "+C"):

a) $\int 2x\,dx$

b) $\int 4t^2\,dt$

c) $\int (3x^5 - 4)dx$

d) $\int \frac{3}{g^4} dg \quad (g \neq 0)$

e) $\int 6\sqrt[5]{p^3}\, dp$

f) $\int \frac{4y-3}{y^{2/3}} dy \qquad (y \neq 0)$

The Definite Integral

A **definite integral** of a function is the difference between the integrals of f(x) at two values of x. Suppose we integrate f(x) and get F(x). Then the integral of f(x) when x = a would be F(a), and the integral when x = b would be F(b).

The definite integral of f(x), with respect to x, between a and b, is written as:

$$\int_a^b f(x)dx = F(b) - F(a) \qquad \text{(where b > a)}$$

For example, the integral of f(x) = $2x^2 - 4$ between the values x = -3 and x = 5 is written as $\int_{-3}^{5}(2x^2-4)\,dx$ and reads "the integral from -3 to 5 of $2x^2$ - 4 with respect to x".

Note: definite integrals do NOT include the constant of integration!

$$\int_a^b f(x) = [F(b)+C] - [F(a)+C] = F(b) - F(a)$$

Example 2: Evaluate $\int_{-1}^{3}(2x-1)\,dx$

To find a definite integral:

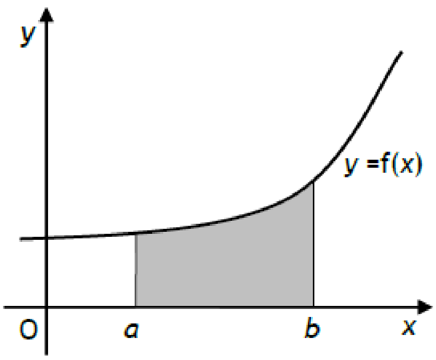
- prepare the function for integration
- integrate as normal, but write inside square brackets with the limits to the right
- sub each limit into the integral, and subtract the integral with the lower limit from the one with the higher limit

Example 3: Evaluate $\int_0^2 (p+1)(p-1)\,dp$

Example 4: Evaluate $\int_1^{\sqrt{3}} (x^2 - 2x)\,dx$

Example 5: Find the value of g such that $\int_{-2}^{g}(6x+5)\,dx = 6$.

Area Between a Curve and the x - axis.



In the diagram opposite, the area of the shaded section can be obtained by finding the area under the graph from 0 to b, and subtracting the area from 0 to a.

The value of each of these areas can be determined by integrating the function and substituting b or a respectively.

The area enclosed by the curve $y = f(x)$, the lines $y = a$, $y = b$ and the x - axis is equal to the definite integral of $f(x)$ between a and b

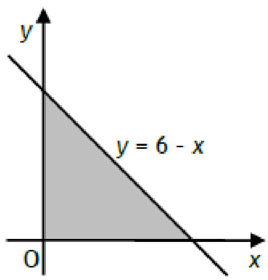
i.e. Area = $\int_a^b f(x)\,dx$

Example 6: For each graph below,

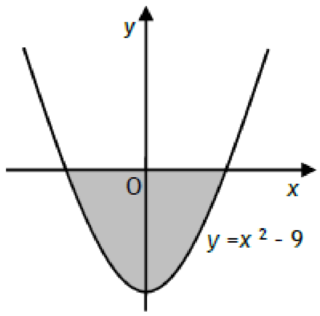
(i) write down the integrals which describe the shaded regions

(ii) calculate the area of the shaded region

a)



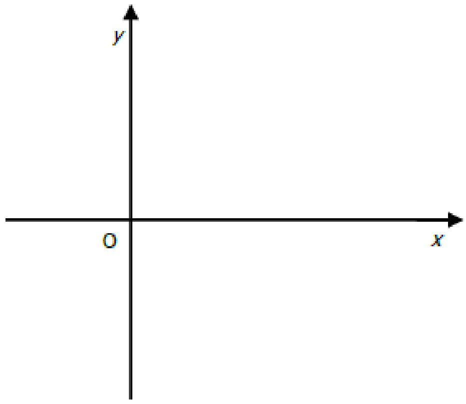
b)



Example 7:

a) Evaluate $\int_{-1}^{7}(2x-6)\,dx$

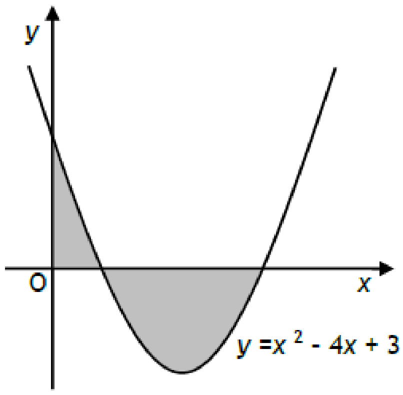
b) (i) Sketch below the area described by the integral $\int_{-1}^{7}(2x-6)\,dx$.



To find the area between a curve and the x-axis:

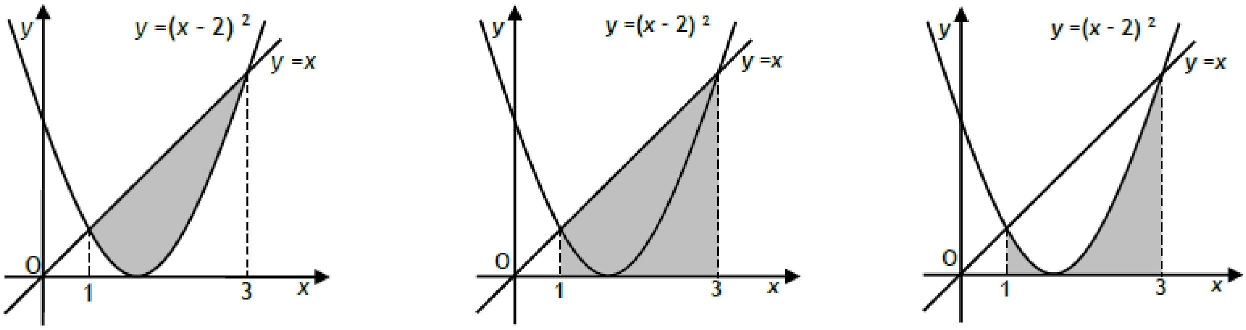
1. Determine the limits which describe the sections above and below the axis
2. Calculate areas separately
3. Find the total, ***IGNORING THE NEGATIVE VALUE OF THE SECTION BELOW THE AXIS.***

Example 8: Determine the area of the regions bounded by the curve $y = x^2 - 4x + 3$ and the x - and y - axes.



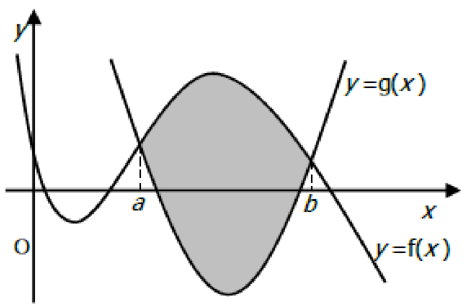
Area Between Two Curves

Consider the area bounded by the curves $y = (x - 2)^2$ and $y = x$.



Area $= \int_1^3 x \, dx - \int_1^3 (x-2)^2 \, dx$

The diagrams above show that the area between the curves is equal to the area between the top function (x) and the x - axis MINUS the area between the bottom curve ($(x - 2)^2$) and the x - axis.



The area between the curves $y = f(x)$ and $y = g(x)$ (which meet at the points where $x = a$ and $x = b$) is given by:

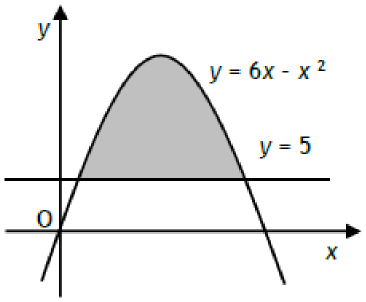
$$A = \int_a^b (f(x) - g(x))\,dx$$

where:

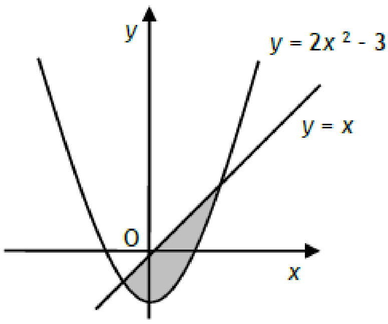
- $f(x)$ is the TOP function and $g(x)$ is the BOTTOM
- $b > a$

Example 9: Write down the integrals used to determine the areas shown below:

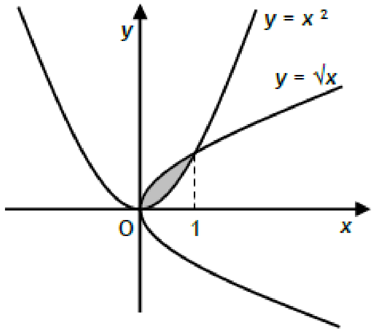
a)



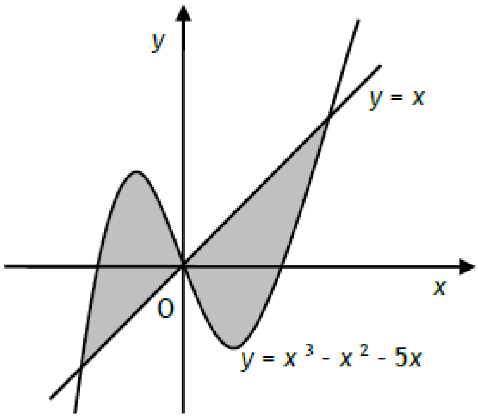
b)



c)

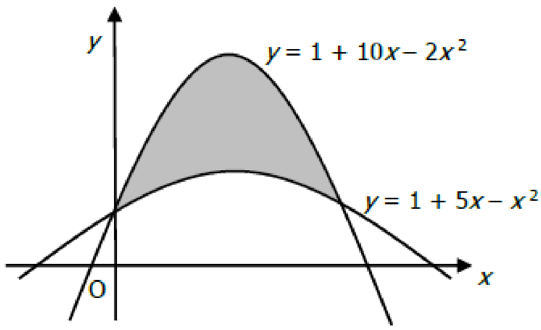


Example 10: Find the area enclosed between the curve $y = x^3 - x^2 - 5x$ and the line $y = x$



Past Paper Example 1: Evaluate $\int_{1}^{9} \frac{x+1}{\sqrt{x}}\,dx$

Past Paper Example 2: Find area enclosed between the curves $y = 1 + 10x - 2x^2$ and $y = 1 + 5x - x^2$.



Past Paper Example 3: The parabola shown in the diagram has equation

$$y = 32 - 2x^2.$$

The shaded area lies between the lines $y = 14$ and $y = 24$

Calculate the shaded area.