Differentiation

1. $f(x) = 3x^3 - 4x$. Calculate the value of $f'(1)$.

2. $f(x) = (2x - 1)^2$. Find $f'(-2)$

3. $y = 4x^2 - 3x + 5$. Calculate the value of $\frac{dy}{dx}$ when $x = 2$.

4. $y = \frac{x^2 - 1}{x}$. Find the value of $\frac{dy}{dx}$ when $x = 3$.

5. $f(x) = \sqrt{x}(4 + 2\sqrt{x})$. Find $f'(4)$.

6. $f(x) = x^3(x - 1)$. Find the value of $f'(-1)$.

7. $y = \frac{x - 3x^2}{x^3}$. Calculate the value of $\frac{dy}{dx}$ when $x = -2$.

8. $f(x) = \left(x + \frac{1}{x}\right)^2$. Find $f'(½)$.

9. $f(x) = \frac{x^2 - 2x}{\sqrt{x}}$. Calculate $f'(16)$.

10. $y = \frac{x^3 - 6x}{x\sqrt{x}}$. Find the value of $\frac{dy}{dx}$ when $x = 4$.

11. $f(x) = \frac{\sqrt{x} + x}{x^2}$. Find $f'(1)$

12. Find the rate of change of $y = 6x - 2x^2$ at $x = 2$.

13. Find the rate of change of $y = \frac{1 - 4x}{x^2}$ at $x = -2$.

14. $f(x) = x(3x - 1)^2$. Find the gradient of the tangent to this curve at $x = -1$.

15. $f(x) = \frac{x - 3}{x^2\sqrt{x}}$. Find the gradient of the tangent to $f(x)$ at the point where $x = 1$.

16. The distance, d metres, travelled on a fairground ride is calculated using the formula $d(t) = 8t^2 - 4t$, where t is the time in seconds after the start of the ride. Calculate the speed of the ride after 3 seconds.

17. The height, h, of a ball thrown upwards is calculated using the formula $h(t) = 30t - 2t^2$, where t is the time in seconds after the ball is thrown.
    Calculate the rate of change in the height of the ball after
    (a) 5 seconds (b) 7.5 seconds. Explain your answer.

Differentiation – 2

1. Differentiate
   (a) $y = 3x^4 - 4x^2 + 2x$ (b) $f(x) = x^2(2x^3 - x)$ (c) $f(x) = 3(4x - 1)(x + 2)$

   (d) $y = \sqrt{x}(x - 4)$ (e) $f(x) = \dfrac{x^3 + 3x - 1}{x^2}$ (f) $\dfrac{3x^3 + x}{\sqrt{x}}$

2. $y = x^2(x - \sqrt{x}\,)$. Find $f'(4)$.

3. Given $f(x) = \dfrac{2x}{\sqrt[3]{x}} + x^3$, find $f'(8)$.

4. Given $y = 3x - \dfrac{1}{x^2}$. Find the rate of change when $x = 2$.

5. The distance a rocket travels is calculated using the formula $d(t) = 4t^3$, where t is the time in seconds after lift-off.
   Calculate the speed of the rocket after 8 seconds.

6. Find the equation of the tangent to the curve $y = 3x^3 - 4x + 1$ at the point (1,0).

7. Find the equation of the tangent to the curve $y = \dfrac{4\sqrt{x}}{x} + 2x$ at the point where $x = 4$

8. A curve has equation $y = 3x^2 - 9x + 1$. A tangent to this curve has gradient 3. Find the equation of this tangent.

9. A curve has equation $y = x^2 + 5x + 7$. A tangent to this curve meets the positive direction of the x-axis at 45°. Find the equation of this tangent.

10. A curve has equation $y = \dfrac{x^4}{4} - 32x$. A tangent to this curve is parallel to the x-axis.
    Find the equation of this tangent.

11. Show that the curve $y = x^3 - 6x^2 + 12x - 5$ is never decreasing.

12. Show that the curve $y = 12x^2 - 6x - 8x^3$ is never increasing.

13. Show that the curve $y = x^3 - x^2 + x$ is always increasing.

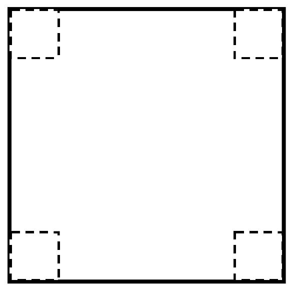
14. Find the intervals in which $y = x^3 - 6x^2 + 5$ is increasing.

15. Find the stationary points of the curve $y = x^3 - 12x + 3$ and determine their nature.

16. A curve has equation $f(x) = x^3 + 4x^2 - 3x - 18$.
    (a) Show that $(x + 3)$ is a factor of $f(x)$.
    (b) Find the points where $f(x)$ cuts the x and y axes.
    (c) Find the stationary points of $f(x)$ and determine their nature.
    (d) Make a sketch of $f(x)$.

17. A curve has equation $f(x) = 8x^3 - 3x^2$
    (a) Find the stationary points of $f(x)$ and determine their nature.
    (b) Find the maximum and minimum values of $f(x)$ in the interval $-2 \le x \le 1$.

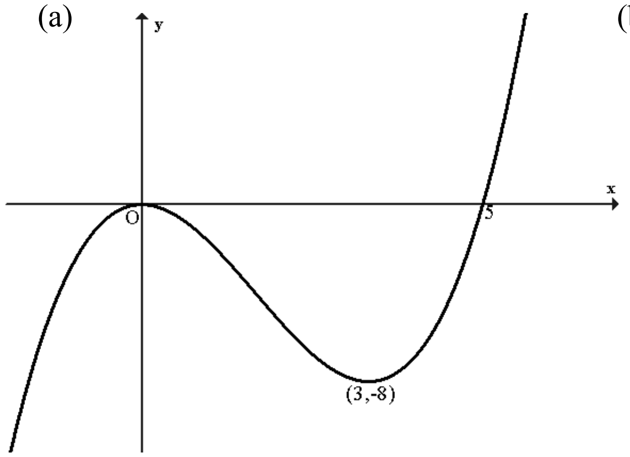
18. A square piece of card of side 20 cm has a square of side x cm cut from each corner.
    An open box is formed by turning up the sides.



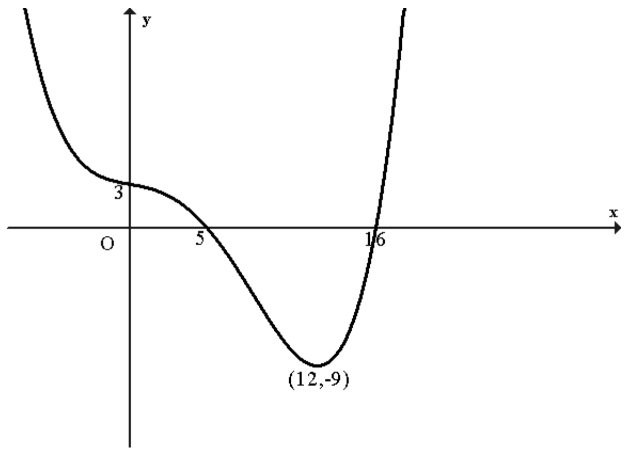
    (a) Show that the volume of the box can be written as $V = 400x - 80x^2 + 4x^3$
    (b) Find the maximum volume of the box.

19. For each function $f(x)$ below, sketch $f'(x)$.

(a)



(b)



20. $y = x^3 - x^2$. Show that $x\dfrac{dy}{dx} - 2y = x^3$

Differentiation – 3

1. Find the derivative of

(a) $y = x^2 + 3\sqrt{x}$ (b) $f(x) = \dfrac{x^2 - 4}{\sqrt{x}}$ (c) $y = \dfrac{(x-2)(x+1)}{\sqrt{x}}$

(d) $y = (4x - 2)^3$ (e) $y = \sqrt{6x - 4}$ (f) $f(x) = \sin 4x$

(g) $y = 2\cos^2 x$ (h) $y = \sin^3 x$

2. The height of a ball projected upwards is calculated using the formula $h(t) = 30t - t^2$, where t is the time in seconds after being projected.

   (a) Find the height of the ball after 10 seconds.
   (b) Find the speed of the ball after 12 seconds.

3. Find the equation of the tangent to the curve $y = x^3 - x^2 - 1$ at the point (2,3).

4. Find the equation of the tangent to the curve $y = 6\sqrt{x} - \dfrac{2}{x^2}$ at the point where $x = 1$.

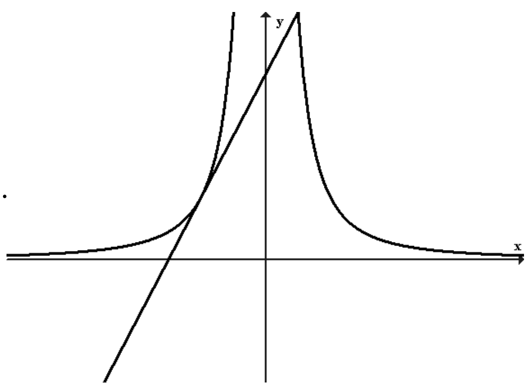
5. Find the equation of the tangent to the curve $y = \sin^2 x$ at the point where $x = \dfrac{\pi}{6}$.

6. A curve has equation $y = (3x - 2)^4$. A tangent to this curve has gradient 12.

   (a) Find the point of contact of the tangent and the curve.
   (b) Find the equation of this tangent.

7. A tangent to the curve $y = \dfrac{4}{x^2}$ is parallel to the line $y = x$.

   Find the equation of this tangent.



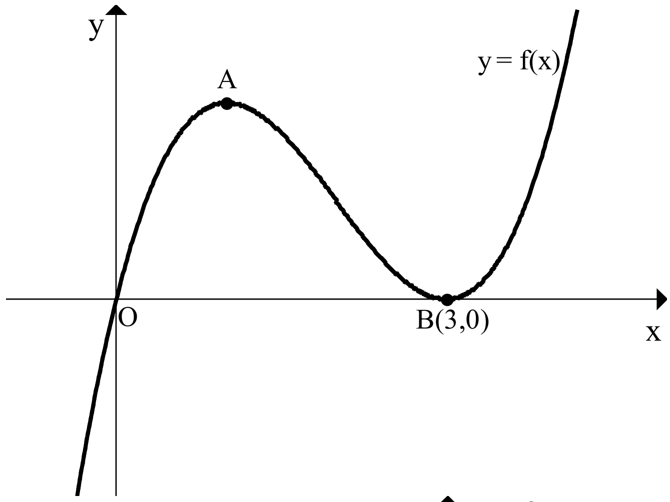
8. Show that the function $f(x) = 6x^2 - x^3 - 12x$ is never increasing.

9. Show that $y = x^3 + 4x + 1$ is always increasing.

10. Find the values of x for which $y = x^3 + 6x^2 - 36x$ is increasing.

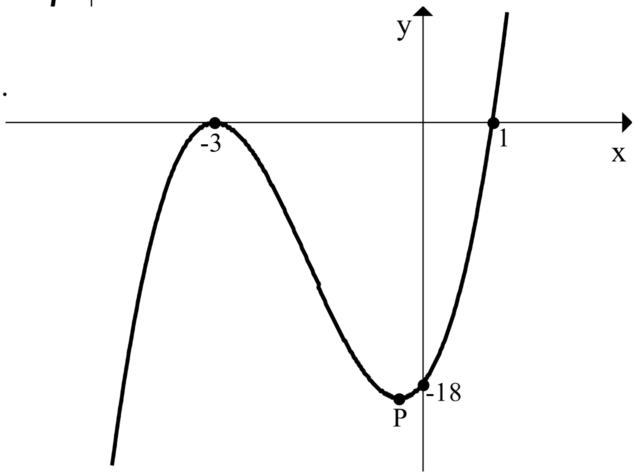
11. Find the values of x for which $y = x^3 + 3x^2 - 9x + 1$ is decreasing.

12. A sketch of the graph of y = f(x) where $f(x) = x^3 - 6x^2 + 9x$ is shown opposite. The graph has a maximum at A and a minimum at (3,0). Find the coordinates of A.



13. The diagram opposite shows the graph of y = f(x).

(a) Find a formula for f(x).
(b) Find the coordinates of the turning point at P.



14. A curve has equation $y = x^3 - 3x^2$.

(a) Find where this curve cuts the x and y axes.
(b) Find the stationary points of the curve and determine their nature.
(c) Sketch the curve.

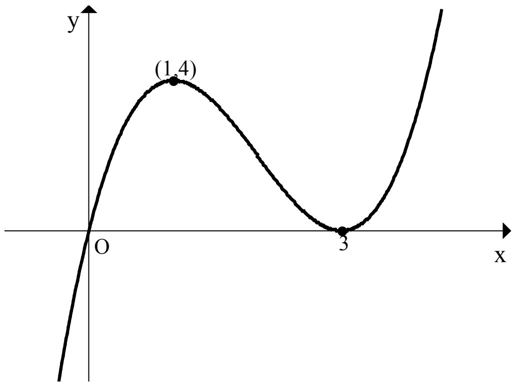
15. Find the minimum and maximum values of $y = 8x^3 - 3x^2$ in the interval $-2 \leq x \leq 1$.

16. Show that the curve $f(x) = x^3 - 4x^2 + 7x$ has no stationary points.

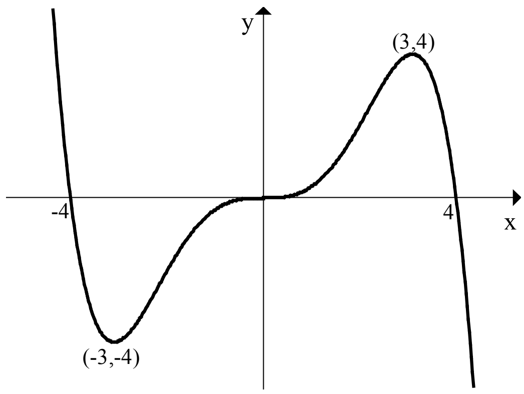
17. Show that the curve $y = \frac{1}{2}x^4 + x^2 - 20x + 15$ has a single stationary point at the point (2, -13).

18. In each example below sketch the graph of $y = f'(x)$.

(a)



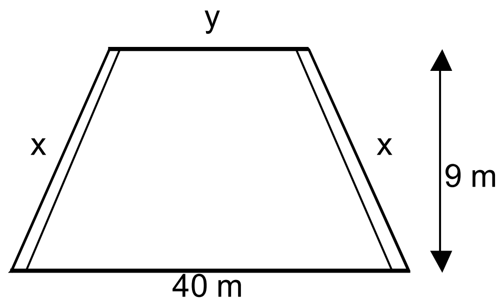
(b)



19. Find the coordinates of the points where the curves $y = x^3 + 2x^2 - 8x$ and $y = x^3 + x^2 + 2x$ have the same gradient.

20. $y = x^2 - 4x$. Show that $\left(\frac{dy}{dx}\right)^2 - 4y - 16 = 0$.

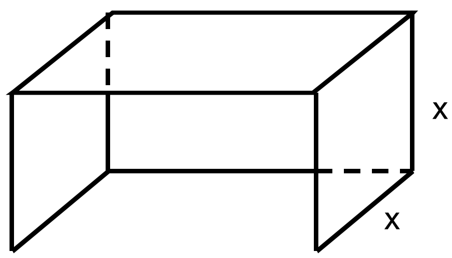
21. The diagram shows the end view of an aircraft hangar. The sloping sides and roof of the hangar are reinforced with metal beams.
The roof beam is of length y metres and there are 2 beams of length x metres at each sloping side.



(a) Show that $y = 40 - 2(x^2 - 81)^{\frac{1}{2}}$

(b) The length of metal needed for the supporting beams is $L = 4x + y$. Find the value of x which minimises this length.

22. A wind shelter, as shown opposite, has a back, top and two square sides. The total amount of canvas used in the shelter is 96 m^2 and the length of each square side is x metres.



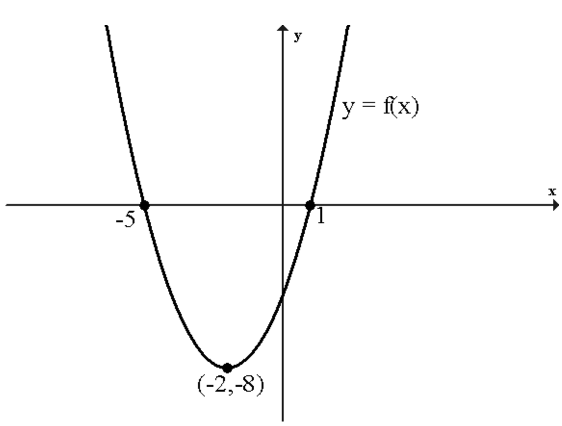
(a) If the volume of the shelter is V cm^3, show that $V = x(48 - x^2)$.

(b) Find the dimensions of the shelter which give a maximum volume.

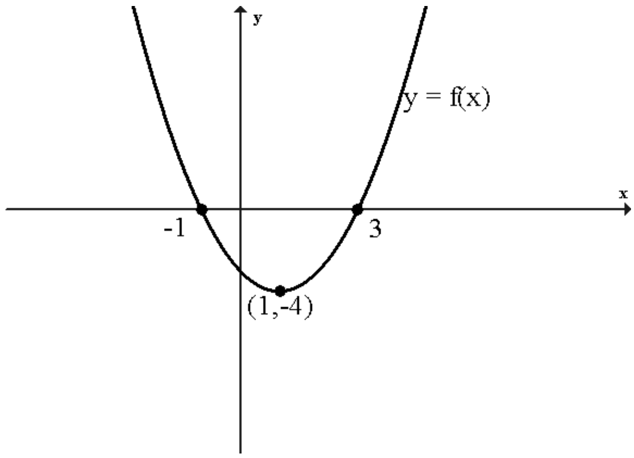
Drawing $f^{/}(x)$

In each question below $y = f(x)$ is drawn. Sketch $f^{/}(x)$ in each case.

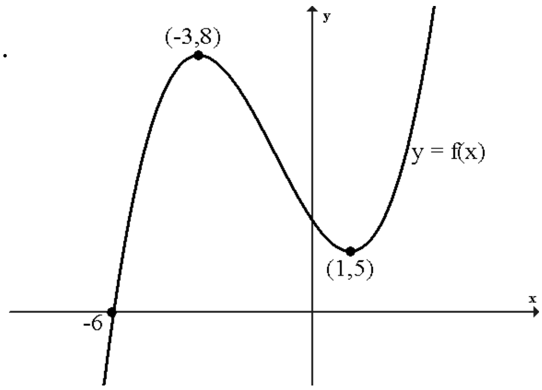
1.



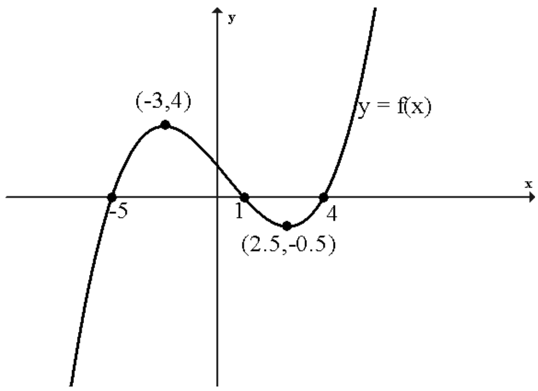
2.



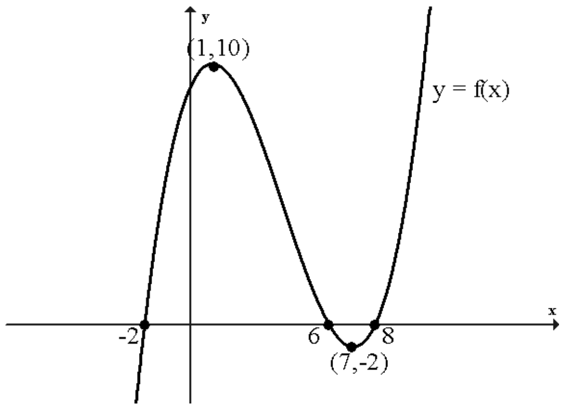
3.



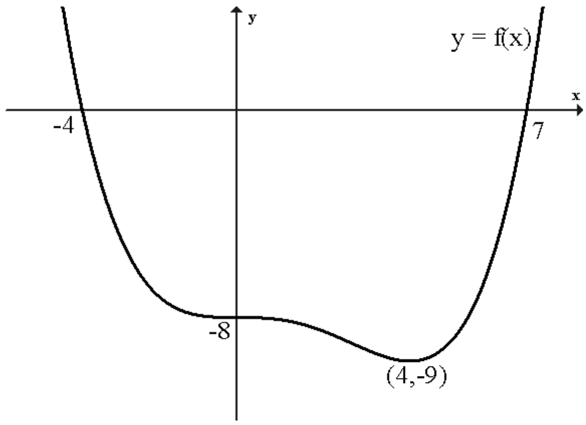
4.



5.



6.



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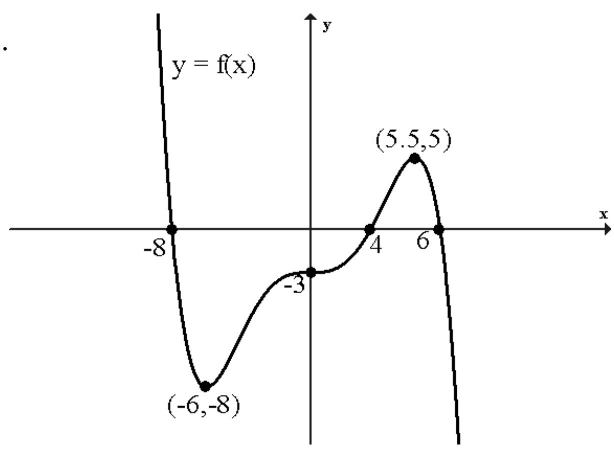
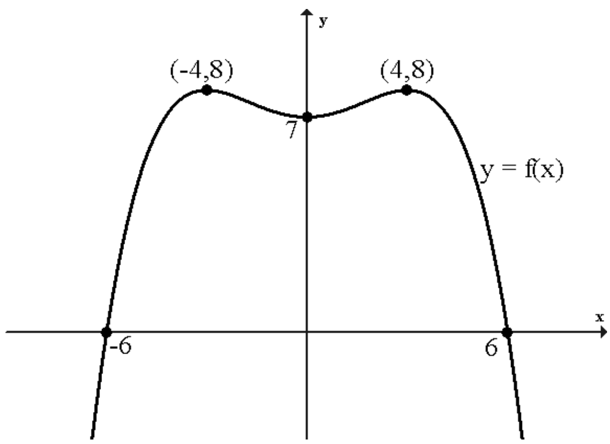
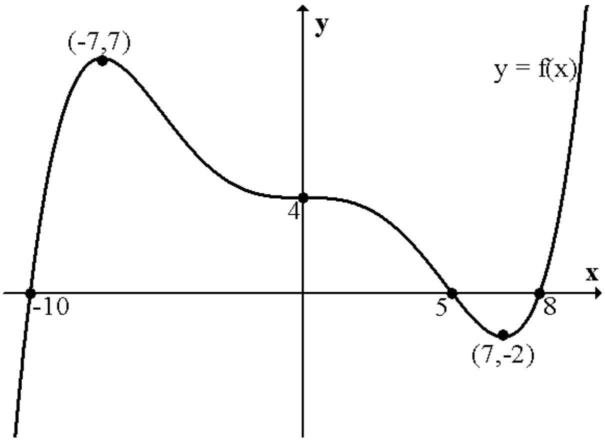


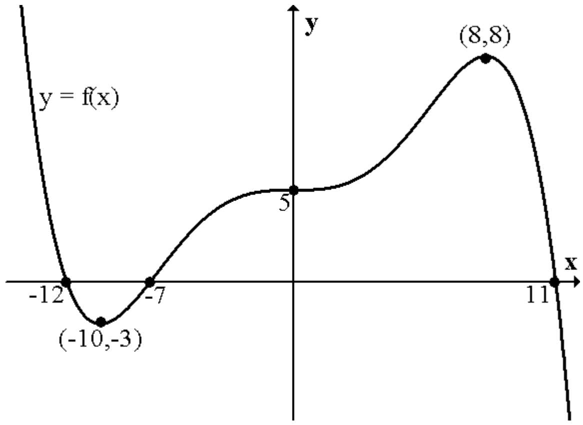
8.



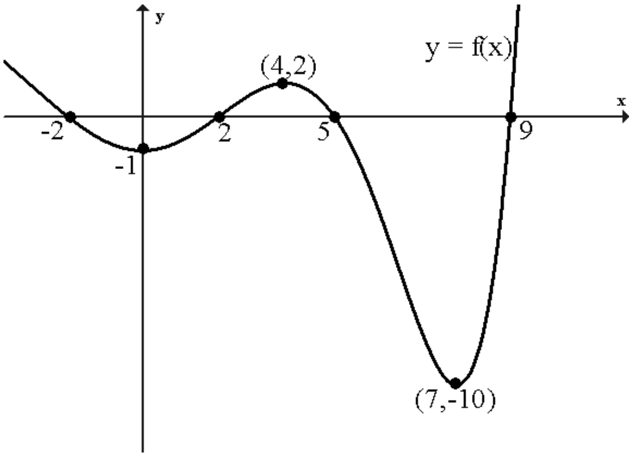
9.



10.



11.



12.

Differentiation
Products and Quotients

1. $y = (2x - 1)(3x + 2)$. Calculate $\frac{dy}{dx}$ when $x = 2$.

2. $f(x) = \frac{x^3 - 2x^2}{x}$. Calculate $f^{/}(-3)$.

3. $f(x) = \left(2x - \frac{4}{x}\right)^2$. Calculate the value of $f^{/}(-2)$.

4. $y = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2$. Calculate $\frac{dy}{dx}$ when $x = 3$.

5. $f(x) = \frac{x^3 - 1}{\sqrt{x}}$. Calculate the value of $f^{/}(4)$.

6. $y = \frac{2x - x^2}{\sqrt[3]{x}}$. Calculate $\frac{dy}{dx}$ when $x = 8$.

7. $f(x) = \frac{4}{x^2} + x\sqrt{x}$. Find the value of $f^{/}(4)$.

8. $f(x) = x^4 - \frac{16}{\sqrt{x}}$. Calculate $f^{/}(1)$.

9. $y = \frac{\sqrt{x} - x}{x^2}$. Calculate the value of $\frac{dy}{dx}$ when $x = 4$.

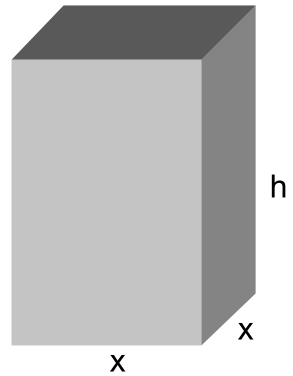
10. $f(x) = \frac{(x^2 + 1)^2}{\sqrt{x}}$. Find $f^{/}(1)$.

11. $f(x) = \frac{x^3 - 4x}{x^2\sqrt{x}}$. Find the value of $f^{/}(4)$

12. $y = \frac{x^2 - x}{\sqrt[4]{x^3}}$. Find $\frac{dy}{dx}$ when $x = 16$.

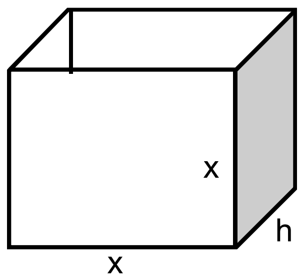
Maxima and Minima

1. A solid cuboid measures x units by x units by h units. The volume of this cuboid is 125 units^3.



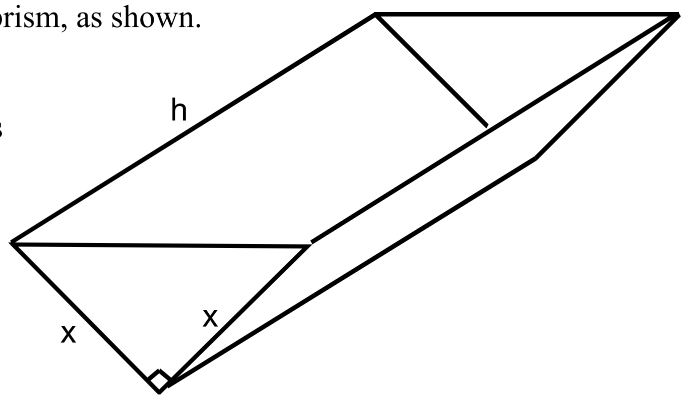
   (a) Show that $h = \frac{125}{x^2}$
   (b) Show that the surface area of this cuboid is given by $A(x) = 2x^2 + \frac{500}{x}$.
   (c) Find the value of x such that the surface area is minimised.

2. An open cuboid (i.e no top) has measurements x units by x units by h units. Its volume is 288 units^3.



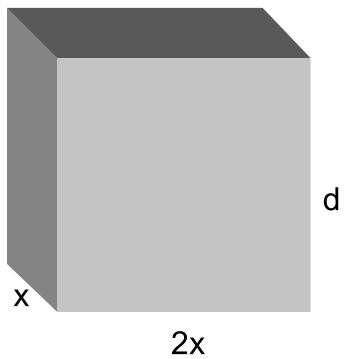
   (a) Show that the surface area of this cuboid is $A(x) = 2x^2 + \frac{864}{x}$.
   (b) Find the dimensions of this cuboid if the surface area is to be minimised.

3. An open trough is in the shape of a triangular prism, as shown. The trough has a capacity of 256 000 cm^3.



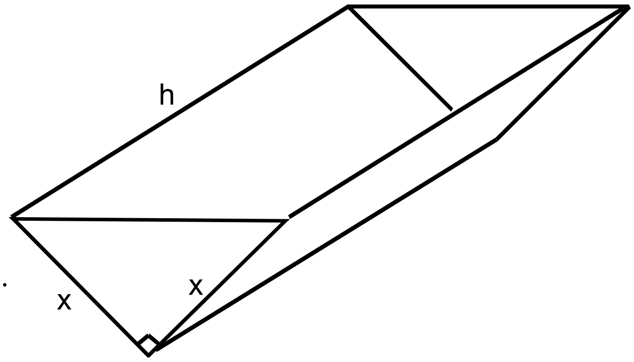
   (a) Show that the surface area of the trough is $A(x) = x^2 + \frac{1\,024\,000}{x}$
   (b) Find the value of x such that this surface area is as small as possible.

4. The diagram shows a solid cuboid. The surface area of this cuboid is 600 cm^2.



   (a) Show that $d = \frac{100}{x} - \frac{2x}{3}$
   (b) Show that the volume of the cuboid is given by $V = 200x - \frac{4}{3}x^3$.
   (c) Find the value of x which maximises this volume.

5. An open trough, as shown, has surface area 4 m^2.



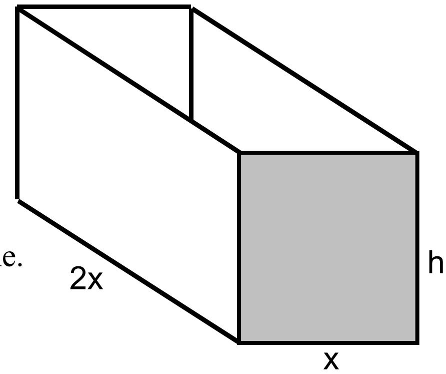
(a) Show that $h = \frac{2}{x} - \frac{x}{2}$.

(b) Show that the volume of the trough is

$$V = x - \frac{x^3}{4}.$$

(c) Show that for a maximum volume $x = \frac{2}{\sqrt{3}}$.

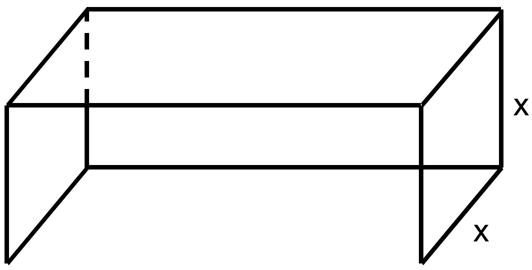
6. An open open cuboid is shown opposite.
The surface area of this cuboid is 12 units^2.



(a) Show that the volume, V units^3, of the cuboid is given by $V(x) = \frac{2}{3}x(6 - x^2)$

(b) Find the exact value of x which gives a maximum volume.

7. A wind shelter, as shown, has a back, top and two square sides.
The total amount of canvas used to make the shelter is 96 m^2.

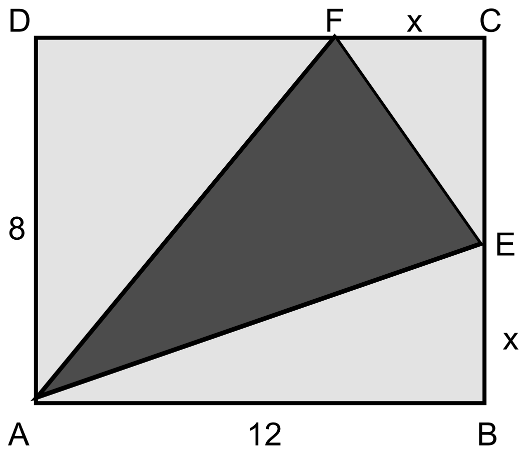


(a) Show that the volume of the shelter is

$$V = x(48 - x^2)$$

(b) Find the dimensions of the shelter of maximum volume, justifying your answer.

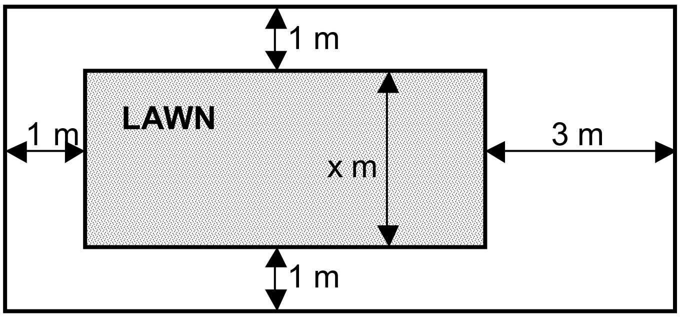
8. A flag consists of red triangle on a yellow rectangular background. In the yellow rectangle ABCD, AB = 12 and AD = 8. EB = CF = x.



(a) Show that the area H, of the red triangle is Given by $H(x) = 48 - 6x + \frac{1}{2}x^2$

(b) Find the biggest possible area of the triangle.

9. A rectangular garden is laid out as shown in the diagram with a rectangular lawn of area 50 square metres surrounded by a border. The lawn has breadth x metres.



(a) If the total area of the garden is A square metres, show that $A = 58 + 4x + \dfrac{100}{x}$.

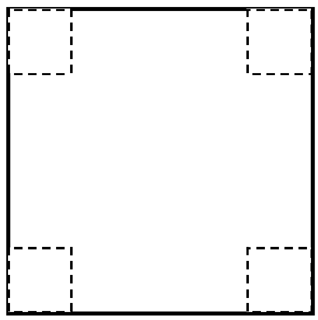
(b) Find the value of x which minimises the area of the garden.

10. When a ship is travelling at a speed of v kilometres per hour it uses fuel at a rate of $(1 + 0.0005v^3)$ tonnes per hour.

(a) Prove that the total amount of fuel used on a voyage of 5000 km at a speed of v kilometres per hour is $A = \dfrac{5000}{v} + 2.5v^2$ tonnes.

(b) Find the speed which minimises the amount of fuel used and the amount of fuel used at this speed.

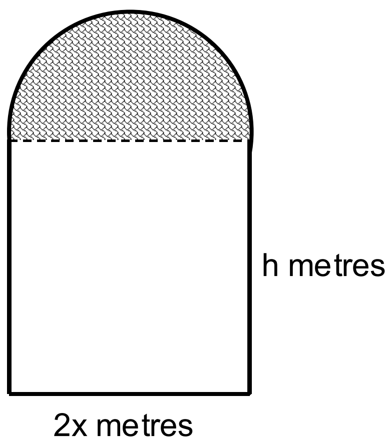
11. A square piece of card of side 50 cm has a square of side x cm cut from each corner. An open box is formed by turning up the sides.



(a) Show that the volume, V, of the open box is given by $V = 2500x - 200x^2 + 4x^3$.

(b) Find the maximum volume of the box, justifying your answer.

12. A window is in the shape of a rectangle surmounted by a semi-circle. The glass in the semi-circular part is stained glass which lets in one unit of light per m^2 and the glass in the rectangular part is clear which lets in 2 units of light per m^2.



(a) If the perimeter of the window is 10 metres, show that $h = \dfrac{10 - 2x - \pi x}{2}$.

(b) Hence show that the amount of light, L, let in by the window is $L = 20x - 4x^2 - \frac{3}{2}\pi x^2$.

(c) Find the value of x such that the design of the window lets in the maximum amount of light.